Ascending Unit Demand Auctions with Budget Limits

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Abstract

We show that the the unit demand auction introduced by Demange, Gale and Sotomayor [5] is incentive compatible even when bidders have budget constraints. Furthermore we show that myopic bidding is an ex post equilibrium. Finally, we show that any other incentive compatible which always outputs a competitive equilibrium (envy free) must coincide with the DGS auction.

1 Introduction

In a unit demand auction each bidder is interested in at most one item. Shapley and Shubik [10] showed that there exists a competitive equilibrium in this setting. Moreover, they showed that among all competitive prices there is a unique vector of competitive prices which is minimal, i.e. pointwise smaller than every other vector of competitive prices. Demange and Gale [4] further showed that the direct auction that outputs a competitive equilibrium with the minimal competitive prices is incentive compatible. Demange et. al [5] introduced a dynamic ascending auction, which we refer to by the DGS auction[1], which obtains the competitive equilibrium with the minimal price vector.

The works mentioned above do not assume that bidders have any budget constraints. In fact most auction theory ignores this issue. It is often the case that bidders have an upper bound on what they can or are willing to pay the auctioneer. For instance, in online ad auctions, advertisers are asked to submit both how much they value each impression, and an upper bound on the amount they are willing to spend.

Only recently several studies incorporated budgets constraints into their settings. Since utilities are no longer quasi linear, this change requires different analysis techniques, which give rise to different results. Therefore, studying budget constraints is important both conceptually and technically.

This paper generalizes the unit demand auction by letting bidders also have a limited budget. We assume both budgets and values are known only to the bidder. We show that if the values and budgets do not satisfy a simple independence assumption (including ties) a competitive equilibrium might not exist.[2] Under the independence assumption we show that a competitive equilibrium

[1] In [5] this auction is referred to as the *the exact auction mechanism*.

[2] It is well-known that in the case of budget ties (see Che and Gale [3] and Krishna [8]) that the second price auction with budget constraints is not truthful. In this auction the winner is the bidder with the maximal "bid" $min(v_i, b_i)$ and he and pays the second highest bid. We show that a competitive equilibrium might not exist even when there are no budget ties.

always exist and that the direct unit demand auction that outputs a competitive equilibrium with the minimal competitive prices is incentive compatible.

Further we show that in the DGS auction with budgets constraint bidders (not the direct version), myopic bidding is an ex post equilibrium, which is equivalent to say that if every bidder uses a proxy bidder then it is a dominant strategy for i also to use a proxy bidder.[3] Finally we show that any other unit demand auction with budgets constrained bidders which is incentive compatible and outputs a competitive equilibrium must output the one with the minimal competitive prices.

1.1 Related Literature

Roth and Sotomayor [9] consider the setting of two sided matching without contracts. They show that every mechanism which produces a stable matching, in which it is dominant for the men to be truthful, must output the men optimal stable outcome. That is, it outputs the stable outcome produced by the Deferred Acceptance algorithm in which the men propose. Thus, our uniqueness result can be viewed as a generalization of their uniqueness result.

Building on their work, Hatfiled and Milgrom [7] give a theoretical framework that unifies unit demand auctions and two sided matching, called "matching with contracts". In their setting the two sides are hospitals and doctors, and each doctor can have a contract with at most one hospital, where a contract includes the wage the hospital pays the doctor. Their setting reduces to ours by fixing the hospitals' preferences, and assuming each hospital prefers paying as least as possible. Hatfiled and Milgrom show that given the hospitals preferences the (generalized) Deferred Acceptance algorithm outputs a stable outcome (interpreted in our setting as a competitive equilibrium) and it is a dominant strategy for every doctor to state his true preferences. An important assumption in their setting is that the preferences are strict. In the context of an auction, this assumption is being violated in two different ways. First, the items are indifferent about which buyer buys them - in contrast to the strict preference hospitals have over doctors. Moreover, if buyer x values item y for z, he is indifferent between paying z and receiving the item, and not receiving this item at all. Assuming that a buyer buys in the case of indifference is also not enough, as there could be multiple deals which are just as good for the buyer.

We elucidate the problem, by giving examples where the budgets are different, but this problem occurs, and indeed no competitive equilibrium exists. We then show an independence condition, which ensures that this can not happen in our mechanism. Finally, we sketch how to change the mechanism such that the independence condition holds with probability 1.

Finally, in a recent paper Dobzinski et. al [6] showed that there is no incentive compatible Pareto optimal multi unit auction. Roughly speaking, in our setting a competitive equilibrium implies that the outcome is Pareto optimal. Hence our results draw the borderline between possibility and impossibility implementation when bidders have budget constraints, as the unit demand setting is the richest well-known setting in which it is possible to implement a Pareto efficient outcome in dominant strategies when bidders also have budget constraints.

We note, that a corollary of our results is an envy free mechanism for multi-unit auctions. To see this, simply decide before hand to bundle the items into k chunks (of different size), and sell the chunks subject to the constraint that a player can get at most one chunk. One can show that a proper choice of the chunk sizes can guarantee a logarithmic fraction of the optimal revenue.

To summarize, our main contributions are (i) uniqueness - the stable outcome produced by the (generalized) deferred acceptance is the the only possible outcome for stable and incentive

[3] In [1] the authors describe an incentive compatible auction for the special case of position auctions in which this result does not hold.

compatible (ii) showing that in the dynamic setting myopic bidding is an ex post equilibrium and mechanisms, and (iii) putting forward an independence assumption, which does not arise when there are no budgets and generalizes ties.

The paper is organized as follows. In Section 2 we provide the general setting. Incentives results of the DGS auction are given in Section 3, and the uniqueness result is given in Section 4.

2 Preliminaries

In a unit-demand auction there is a finite set of k items and a finite set of n bidders N where each bidder is interested in receiving at most one item. We assume that $n \geq k \geq 1$. Every bidder i has a private valuation vector $v_i = (v_i(x))_{x\in K}$ where $v_i(x) \geq 0$ denotes bidder i's value for item x. In addition every bidder i has a private budget $b_i > 0$; bidder i's payment is strictly less than b_i.[4] A pair $t_i = (v_i, b_i)$ is called a *type*. It is convenient to add a null item, denoted by ϕ, in which its value for each bidder is zero. We assume that any bidder that does not get an item in K gets the null item and pays zero.

In the absence of budget constraints, bidder i's utility from receiving item x and paying p_i is equal to $v_i(x) - p_i$. However, as budgets are incorporated in our model, we assume the utility function for bidder i with type $t_i = (v_i, b_i)$ is given by

$$u((v_i, b_i), x, p_i) = \begin{cases} v_i(x) - p_i & b_i > p_i \\ -1 & b_i \leq p_i \end{cases} \tag{1}$$

where the negative utility for the case $p_i \geq b_i$ can be thought of as bidder i will not complete the transaction if he is required to pay b_i or more.[5]

An *assignment* is a tuple $\mathbf{s} = (s_i)_{i\in N}$ where $s_i \in K \cup \{\phi\}$ such that for every pair of bidders $i, j \in N$ if $s_i, s_j \in K$ then $s_i \neq s_j$. An *outcome* in the auction is a tuple $(s_i, p_i)_{i\in N}$ where $(s_i)_{i\in N}$ is an assignment and p_i is the payment for bidder i.

For simplicity we assume the seller has a reserve price 0 for each item. Note that at this point nothing has been said about the rules of the auction, in particular what are the possible strategies and how the outcome is determined.

Throughout this paper we assume that all values and budgets are integers; similar results can be obtained in the case of general budgets and valuations.

3 The DGS Ascending Auction

In this section we describe and analyze the ascending auction described by Demange et. al [5] generalized to budget constrained bidders. At each stage in the auction the auctioneer holds a vector of prices $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{R}^K_+$ where q_x is the price for item x at stage r.

At the first stage the prices are $\mathbf{q} = (0, \ldots, 0)$, and every bidder submits a subset of items which he is interested in. We refer to this subset as a *demand set*.[6] We say that a subset of items (of K) is *overdemanded* if the number of bidders interested in/demanding **only** items in this set is greater than the number of items in the set. If there is no overdemanded set (with respect to the submitted demand sets) then it is possible to assign each item to a bidder who demands it and the auction is over; in this case if item $x \in K$ is assigned to bidder i he pays q_x and if i is assigned the null object

[4] We do not allow b_i to be a feasible payment just for mathematical convenience.

[5] Replacing this with any other negative utility does not alter the results.

[6] We do not assume here that the demand set i submits, necessarily maximizes i's utility.

he pays zero. Otherwise the auctioneer computes a minimal overdemanded set (with respect to the submitted subsets of the bidders) and for each item in this set it raises the price by one unit. Again, each bidder announces a demand set at the new prices and this process goes on. Eventually the auctioneer will find a possible assignment, since the prices are raised by a unit at each stage.

3.1 Competitive Prices

Denote by $D(\mathbf{q},(v_i,b_i))$ the *true demand set* of a bidder at prices $\mathbf{q}$ when his type is (v_i,b_i), that is

$$D(\mathbf{q},(v_i,b_i)) = \{x \in K \cup \phi | x \in \arg\max_{y \in K \cup \{\phi\}} \{v_i(y) - q_y : q_y < b_i\}\}. \tag{2}$$

Let $\mathbf{t} = ((v_1,b_1),\ldots,(v_n,b_n))$ be a profile of types. A vector of prices $\mathbf{q}$ is *competitive* (with respect to $\mathbf{t}$) if there is an assignment $\mathbf{s} = (s_i)_{i \in N}$ such that $s_i \in D(\mathbf{q},(v_i,b_i))$. Such an assignment is said to be *valid* for $\mathbf{q}$.

The following theorem given in [5] (without budget constraints) shows that if all bidders always announce their true demand set, i.e. each bidder demands all the items that maximize his utility under the given prices, then the auction terminates at the minimal competitive price vector. Formally,

Theorem 3.1. *Let $\mathbf{t} = ((v_1,b_1),\ldots,(v_n,b_n))$ be the profile of types. Let $\mathbf{q}^r$ be the prices at stage r and let $\mathbf{q}$ be the price vector at the end of the auction. If at every stage r, each bidder submits his true demand set $D(\mathbf{q}^r,(v_i,b_i))$, then $\mathbf{q}$ is competitive and for any other competitive price vector $\tilde{\mathbf{q}}$, $\mathbf{q} \leq \tilde{\mathbf{q}}$.*

The proof of Theorem 3.1 is identical to the proof of Theorem 1 by Demange et. al in [5] and is therefore omitted. Their proof uses the celebrated Hall theorem which asserts that a possible allocation exists if and only if there is no overdemanded set. Roughly speaking, the proof of [5] remains in the "abstract level" of demand sets, and therefore the presence of budgets does not change any of their arguments.

3.2 Competitive Equilibrium

A tuple $(\mathbf{q},\mathbf{s})$ is called a *competitive equilibrium* if $\mathbf{s}$ is valid for $\mathbf{q}$, and in addition for any item $x \in K$, if $s_i \neq x$ for every every bidder i, then $q_x = 0$. In other words the price of non allocated items in equilibrium is zero.

In [5] Demange et. al also show that for every competitive prices, there exists an assignment that together form a competitive equilibrium. Interestingly, as illustrated in the following simple example, this is not true in our context.[7]

Example 3.2. *Consider one item x and two bidders 1 and 2. Let $b_1 = b_2 = 10$ and $v_1 = 15$, $v_2 = 20$. For any $q_x < 10$ both bidders' true demand set contains x. Therefore any competitive price is at least 10, but in any such price the item is not allocated.*

Example 3.2 shows that if ties are allowed then there is no competitive equilibrium. The following example shows that a competitive equilibrium does not exist even with no ties.

Example 3.3. *Consider two items, x and y and three bidders 1, 2 and 3. Let $b_1 = 10$, $b_2 = 11$ and $b_3 = 1000$. Let $v_1(x) = 1500$, $v_2(y) = 2000$, $v_3(x) = 20$, $v_3(y) = 21$ and all other values are zero. Note that the final prices will be $q_x = 10$ and $q_y = 11$, bidder 3 will get either item x or item y, and the other item will not be allocated.*

[7] This is consistent with the assumption of strict preferences in Hatfield and Milgrom [7].

Examples 3.2 and 3.3 motivate the following definition:

Definition 3.4 (Independence). *We say that n numbers $x_1, \ldots, x_n$ are* independent*, if it is not possible to find two different nonempty subsets containing positive numbers, that sum up to the same number. Alternatively, for every linear combination $\sum_{i=1}^{n} e_i x_i$ where $e_i \in \{-1, 0, 1\}$ then if $e_i \neq 0$ then $x_i = 0$.*

For any profile of types $\mathbf{t} = ((v_1, b_1), \ldots, (v_n, b_n))$ we denote by $H(\mathbf{t})$ the set of numbers $b_1, \ldots, b_n, v_1(1), v_1(2), \ldots, v_1(k), v_2(1), \ldots, v_2(k), \ldots, \ldots, v_n(1), \ldots, v_n(k)$.

Independence Assumption: For every type profile $\mathbf{t}$, the numbers in $H(\mathbf{t})$ are independent.[8]

Note that in Example 3.3, the types are not independent, since $v_3(y) + b_1 = v_3(x) + b_2$. We will show:

Theorem 3.5. *Under the independence assumption if $\mathbf{q}$ is the minimal competitive price vector then there exist an assignment $\mathbf{s}$ such that $(\mathbf{q}, \mathbf{s})$ is a competitive equilibrium.*

The proof of Theorem 3.5 is given in Subsection 3.3.1. We first develop a useful tool in the next subsection.

3.3 (Almost) Envy Graphs

In this section we define an almost envy free graph and provide some useful properties. Roughly speaking in such a graph bidder i "points" to bidder j if i would envy j had the price of s_j would have been smaller by one unit. Formally,

Definition 3.6. *Let $\mathbf{t} = ((v_1, b_1), \ldots, (v_n, b_n))$ be a profile of types. Let $\mathbf{q}$ be a minimal competitive price vector w.r.t $\mathbf{t}$ and let $\mathbf{s}$ be a valid assignment for $\mathbf{q}$. In a* $(\mathbf{q}, \mathbf{s})$-graph *or an* almost envy free graph *$T = (V, E)$, the set of nodes is the set of bidders N, and there exist a directed edge $(i, j) \in E$ if and only if decreasing the price by one unit the price of s_j will cause i to envy j, i.e. $u((v_i, b_i), s_i, q_{s_i}) < u((v_i, b_i), s_j, q_{s_j} - 1)$. An edge $(i, j) \in E$ is colored green if $q_{s_j} = b_i$ and red otherwise.*

Intuitively, a green edge from i to j captures bidder i's envy due to his budget limit, and a red edge (i, j) implies that i has the budget for getting j's item in his price but is indifferent to such an outcome, i.e. $v_i(s_i) - p_i = v_i(s_j) - p_j$.

Lemma 3.7. *Let $\mathbf{q}$ be a minimal competitive price vector and let $\mathbf{s}$ be a valid assignment for $\mathbf{q}$. Let T be the $(\mathbf{q}, \mathbf{s})$-graph and let $p_i = q_{s_i}$.*

1. *If $p_i > 0$ then the indegree of i is at least 1.*
2. *If $p_i > 0$, then there exist two vertices j, l such that l is a predecessor of i (possibly $l = i$), j points to l, and either $p_j = 0$ or the edge (j, l) is green.*
3. *T contains no cycles.*
4. *If $p_i > 0$ then $p_i \neq v_i(s_i)$.*

[8] Unless specified otherwise we will have this assumption through the entire paper.

Proof. 1. Let i be such that $p_i > 0$ and suppose the indegree of i is zero. Then the price vector $\tilde{q}$ in which $\tilde{q}_{s_i} = q_i - 1$ and $\tilde{q}_j = q_j$ for all $j \neq i$ is competitive since $\mathbf{s}$ is valid for $\tilde{\mathbf{q}}$. This contradicts the minimality of $\mathbf{q}$.

2. Let i be such that $p_i > 0$ and assume the claim doesn't hold. This implies that there exists a cycle of red edges. Let $i_1, i_2, \ldots, i_m, i_1$ be such cycle. Since for every $l = 1, \ldots, m$ we have $p_{i_l} - p_{i_l+1} = v_i(s_{i_l}) - v_i(s_{i_{l+1}})$ ($l+1$ is taken modulo m) we obtain that

$$0 = p_{i_1} - p_{i_2} + p_{i_2} - p_{i_3} + \cdots + p_{i_m} - p_{i_1} =$$

$$v_{i_1}(s_{i_1}) - v_{i_1}(s_{i_2}) + v_{i_2}(s_{i_2}) - v_{i_2}(s_{i_3}) + \cdots v_{i_m}(s_{i_m}) - v_{i_m}(s_{i_1}),$$

contradicting the independence assumption.

3. We first show that no vertex has in-degree greater than 1. Suppose that i has indegree strictly greater than 1. Let $\alpha \neq \beta$ be two predecessors of i. According to part 2, there exists two paths $j_{\alpha 1}, j_{\alpha 2}, \ldots, j_{\alpha k} = \alpha$ such that either $p_{j_{\alpha 1}} = 0$ or the edge $(p_{j_{\alpha 1}}, p_{j_{\alpha 2}})$ is green, and similarly there exists a path $j_{\beta 1}, j_{\beta 2}, \ldots j_{\beta r} = \beta$ such that either $p_{j_{\beta 1}} = 0$ or the edge $(p_{j_{\beta 1}}, p_{j_{\beta 2}})$ is green. We can now express the payment p_i by using any one of the two paths (which contradicts the independence assumption). For example, if $p_{j_{s1}} = 0$ then

   and if $(p_{j_{\alpha 1}}, p_{j_{\alpha 2}})$ is green then

4. We first show that no node has an indegree larger than 1. Suppose towards a contradiction that i has indegree $i > 0$. By part 2 of the lemma there exist two different predecessors paths $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_r$ where $i_m = j_r = i$ with satisfy the conditions in part 2. We choose these paths such that all edges are red perhaps but the first one, i.e. if the first is green then it is close as possible to i. Consider the first path; either (i_1, i_2) is green or $p_{i_1} = 0$. Note that on the first path for every $2 < l < m$ $p_{i_l+1} = v_{i_l}(s_{i_l}) - v_{i_l}(s_{i_{l+1}}) - p_{i_l}$ and for $l = 2$ either $p_{i_2} = v_{i_1}(s_{i_1}) - v_{i_1}(s_{i_2})$ or $p_{i_2} = b_1$. Since the payments of the second path can be written similarly, this implies that we can express p_i in two different ways only with budgets or values, contradicting the independence assumption.

   Since by the first part of the lemma the maximum indegree is one, to complete the proof it is enough to show that there does not exist a cycle in which all it nodes have exactly outdegree 1. Suppose that there exists such a cycle $i_1, i_2, \ldots, i_m, i_1$. Let $\tilde{\mathbf{q}}$ be the price vector in which for every $j = 1, \ldots, m$ let $\tilde{q}_{s_{i_j}} = q_{s_{i_j}} - 1$. and all other prices remain the same. We claim that $\tilde{\mathbf{q}}$ is a competitive price vector contradicting the minimality of $\mathbf{q}$: let $\tilde{\mathbf{s}}$ be the assignment for every bidder $j = 1, \ldots, m$ $s_{i_j} = s_{i_{j+1}}$ (where $j+1$) is taken modulo m). Note that $\tilde{\mathbf{s}}$ is a valid for $\tilde{\mathbf{q}}$.

5. Suppose $p_i > 0$ and assume $p_i = v_i(s_i)$. By part 2 there exist a path $i_1, i_2, \ldots, i_m$ where $i_m = i$ where either $p_{i_1} = 0$, or $p_{i_1} = b_{i_1}$. Similarly to part 2 we can express p_i in a linear combination of values and/or budgets, except i's. But since p_i also equals $v_i(s_i)$ this contradicts the independence assumption.

□

3.3.1 Proof of Theorem 3.5

Let $\mathbf{q}$ be a minimal competitive price vector and $\mathbf{s}$ a valid assignment for $\mathbf{q}$. We show that $(\mathbf{q}, \mathbf{s})$ is an equilibrium. Suppose there exist an item $x \in K$ such that $q_x > 0$ and no bidder gets this item.

Observe that there exist two different bidders $l, j \in N$ such that for each $i \in \{l, j\}$ either $q_x = b_i$ or $q_x = v_i(s_i) - v_i(x) - p_i$. We first show that for each $i \in \{l, j\}$ there is a linear combination of elements in $H(\mathbf{t})$ that sum up to q_j. Fix some arbitrary $i \in \{l, j\}$. If $q_j = b_i$ we are done. Suppose $q_x = v_i(s_i) - v_i(x) - p_i$. If $p_i = 0$ we are done. If $p_i > 0$ then by part 2 of Lemma 3.7 there exists a simple path $i_1, i_2, \ldots, i_m$ where $i_m = i$, such that either (i_1, i_2) is a green edge or $p_{i_1} = 0$. But this implies the existence of such a linear combination. It remains to show that for each i the linear combination is different as this contradicts the independence assumption. This follows since for each $i \in \{l, j\}$ either $q_x = b_i$ or $-v_i(x)$ appear only in the linear combination we found for i. $\square$

4 Incentive Compatibility

In this section we analyze the incentive properties of the DGS auction. For any stage r let H_r be the history of demand sets of the bidders up to stage r. A bidding strategy for i is a sequence $\tau_i^1, \tau_i^2, \ldots$, such that for each $r \geq 1$, $\tau_i^r : H_r \times \mathbb{R}_+^K \rightarrow 2^{K \cup \{\phi\}}$ maps a history in H_r and a vector of prices to a demand set. Our results do not depend on the histories' structure. Thus, with a slight abuse of notation we write $\tau_i^r(\mathbf{q})$ to denote the demand set i submits at round r under the strategy τ_i, when the price vector is $\mathbf{q}$.

We say that a strategy τ_i for i is *consistent w.r.t to type* (v_i, b_i) if for every price vector $\mathbf{q}$ and every stage r, $\tau_i^r(\mathbf{q}) = D_i(\mathbf{q}, (v_i, b_i))$. A strategy is *consistent* if there exist a type for which it is consistent with it.[9] Thus every consistent strategy is a myopic strategy w.r.t to the the type it is consistent with.

4.1 The Direct DGS Auction

Essentially, by limiting all bidders to use consistent strategies, the auction is a direct revelation mechanism in which each bidder only submit a type and the auctioneer computes the outcome (e.g. by simulating the whole process). We call this auction the *direct DGS auction*. Formally, the direct DGS auction is defined as follows:

- Every bidder i, submits a bid (v_i, b_i).
- Let $\mathbf{t} = ((v_1, b_1), \ldots, (v_n, b_n))$. If $H(\mathbf{t})$ do not satisfy the independence assumption the auction is terminated.
- The auctioneer computes a competitive equilibrium with $(\mathbf{s}, \mathbf{q})$ where $\mathbf{q}$ is a minimal price vector, assigns s_i to bidder i and charges his q_{s_i}.

In the next theorem we show that bidding the direct DGS auction is *incentive compatible* or *truthful*, that is for every bidder it is a dominant strategy to report his true type.

Theorem 4.1. *The direct DGS auction is truthful.*

We will assume w.l.o.g. that all bid profiles discussed in the proof satisfy the independence assumption. Through out the proof we fix some bidder i and fix the submitted types of all bidders but i, these are $\mathbf{t}_{-i} = (t_j)_{j \in N \setminus \{i\}}$. For any type t_i let $\mu(t_i) = (\mathbf{q}(t_i), \mathbf{s}(t_i))$ be the competitive equilibrium when i bids t_i, and let $p_i(t_i) = q_{s_i}(t_i)$ be his payment.

Lemma 4.2. *For any t_i, t_i' in which $s_i(t_i) = s_i(t_i')$ (i is assigned the same item), $p_i(t_i) = p_i(t_i')$ (i pays the same price).*

[9] Consistent strategies can be thought of bidding through a proxy bidder (see e.g. [2]).

Proof. Fix some type $t_i = (v_i, b_i)$ in which bidder i is assigned an item $x \in K$ and let $\hat{t}_i = (\tilde{v}_i, \tilde{b}_i)$ be the type obtained by t_i by letting $\tilde{v}_i(x) = v_i(x)$, $\tilde{v}_i(y) = 0$ for all other items, and $\tilde{b}_i = b_i$. It is enough to show that $s_i(\tilde{t}_i) = s_i(t_i)$ and $p_i(t_i) = p_i(\tilde{t}_i)$: Suppose this is true. Let t_i and t'_i be two different types in which bidder i obtains the same item x but $p_i(t_i) < p_i(t'_i)$. Therefore $p_i(\tilde{t}_i) < p_i(\tilde{t}'_i)$. But $\mathbf{q}(\tilde{t}_i)$ are competitive prices with respect to $(\tilde{t}_i, t_{-i})$ (as $\mathbf{s}(\tilde{t}_i)$ is a valid assignment) contradicting the minimality of $\mathbf{q}(\tilde{t}'_i)$.

In the following sequence of claims we prove that $s_i(\tilde{t}_i) = s_i(t_i)$ and $p_i(t_i) = p_i(\tilde{t}_i)$.

Claim 4.3. *For every item y, $q_y(\tilde{t}_i) \leq q_y(t_i)$.*

Proof. Since $\mathbf{q}(t_i)$ are competitive with respect to $(\tilde{t}_i), t_{-i})$, and the auction outputs the minimal competitive prices this follows. □

Claim 4.4. *$s_i(\tilde{t}_i) = x$, i.e. i is also assigned x when he reports $\tilde{t}_i$.*

Proof. Assume that this is not the case and let $s_i(\tilde{t}_i) = y \neq x$. Since $\tilde{v}_i(y) = 0$ and since by the previous claim $q_x(\tilde{t}_i) \leq q_x(t_i)$ it must be that $q_x(\tilde{t}_i) = q_x(t_i) = v_i(x)$ otherwise this contradicts that $\mathbf{q}(\tilde{t}_i)$ are competitive. But $q_x(t_i) = v_i(x)$ contradicts part 4 of Lemma 3.7. □

For every item $x \in K$ with $q_x > 0$ denote by $w(x)$ the winner of item x, and let $z(x)$ be a bidder such that $(z(x), w(x))$ is an edge in the μ-graph.

For the next claim we need some definitions. Let A denote the set of items in which their prices decreased from the case that i bids t_i to the case that i bids $\tilde{t}_i$. That is

$$A = \{y \in K : q_y(\tilde{t}_i) < q_y(t_i)\}.$$

We also define two functions from K to $K \cup \{\phi\}$.

We let $\delta(y) = z$ if $s_j(t_i) = y$ and $s_j(\tilde{t}_i) = z$. We let $\gamma(y) = z$ if there exist a pair of bidders j, l and an item w, such that $s_j(t_i) = w$, $s_j(\tilde{t}_i) = z$ $s_l(t_i) = l$ and (j, l) is an edge in the $\mu(t_i)$-graph.

Claim 4.5. *If $y \in A$, then $\delta(y) \in A$. Moreover, if $q_y(t_i) > 0$, and $\gamma(y) \neq x$, then $\gamma(y) \in A$.*

Proof. Assume $s_j(t_i) = y$ and $z = \delta(y)$. Since $\mathbf{q}$ are competitive with respect to (t_i, t_{-i}), j does not prefer z at $q_z(t_i)$ to y at $q_y(t_i)$. But this means that if $q_y(\tilde{t}_i) < q_y(t_i)$, and $q_z(\tilde{t}_i) = q_z(t_i)$, then j would prefer y when i submits $\tilde{i}$ contradicting the competitiveness of $\mathbf{q}(\tilde{t}_i)$.

To prove the second part suppose $q_y(t_i) > 0$ and $z = \gamma(y) \neq x$. Let j be such that $s_j(t_i) = w$ and $s_j(\tilde{t}_i) = z$, and l be such that $s_l(t_i) = y$. If (j, l) is red. Such a configuration exists by part 1 of Lemma 3.7. Thus, j is indifferent between getting w in $q_w(t_i)$ and y in $q_y(t_i)$. Therefore if $y \in A$ then j is strictly better off getting y in $q_y(\tilde{t}_i)$ then getting any other item x' in $q_{x'}(t_i)$, implying that $w \in A$. If (j, l) is green, then obtaining y in $q_y(\tilde{t}_i) - 1$ or less is strictly better for j than obtaining any item x' in $q_{x'}(t_i)$, again implying that $y \in A$. □

To finish the proof we need to show that $p_i(t_i) = p_i(\tilde{t}_i)$. Suppose that $p_i(t_i) > p_i(\tilde{t}_i)$. Hence $x \in A$. Therefore since there are no cycles in $\mu(t_i)$-graph, by the last claim it must be that some item whose price is zero when i submits t_i belongs to A - a contradiction. □

By Lemma 4.2 bidder i is given a "menu" of prices, a price for each item which does not depend on his submitted type. To complete the proof of Theorem 4.1 it remains to show that if i submits his true type he obtains the item that maximizes his utility in these prices. Formally,

Lemma 4.6. *Let $t_i = (v_i, b_i)$ be bidder i's type and let $\mathbf{p}^i$ be the prices i is provided, that is q^i_x is the price i will pay if he is assigned item x. Then $s_i(t_i) = \max_{x \in K \cup \{\phi\}} u(t_i, x, p^i_x)$.*

Proof. Fix the types $\mathbf{t}_{-i}$ of all other bidders but i and assume towards a contradiction that item x maximizes i's utility but i prefers item y in the given prices. That is $p^i_y < b_i$ and

$$v_i(y) - p^i_y > v_i(x) - p^i_x.$$

Let $t'_i = (v'_i, b'_i)$ where $b'_i = \infty$,$v_i(y) = \infty$ and for every item $z \neq y$ $v_i(z) = 0$. by the competitiveness of $\mathbf{q}(t'_i)$, i must get item y when the profile of types is (t'_i). Moreover since his payment does not depend on his type $q_y(t'_i) = p^i_y$.

By competitiveness of the prices $\mathbf{q}(t_i)$ it must be that $q_y(t_i) > p^i_y$ (otherwise i would demand y in when his type is t_i). To finish the proof we show that $p^i_y > q_y(t_i)$ which is of course a contradiction.

Let $A = \{z \in K \ |q_x(t_i) > q_x(t'_i)\}$, i.e. A is the set of items which their prices decreased from t_i to t'_i. If bidder j got an item $z \in A$, i.e. $s_j(t_i) = z$, then $s_j(t'_i) \in A$ otherwise j would demand z when i bids t_i. Similarly, if $z \in A$ and (j, i) is en edge in the $\mu(t_i)$-graph then $s_j(t'_i) \in A$.

We showed more than $|A|$ items belong to A which is a contradiction. □

4.2 Ex-Post Equilibrium in the (non-direct) DGS Auction

In this section we show that myopic bidding is an ex post equilibrium in the ascending DGS auction. That is, if all bidders but i use consistent strategies then using the consistent strategy consistent with t_i is a best response for i. In particular this implies that if all bidders but i use consistent strategies then after every stage bidder i is better off submitting his true demand set. In [1] an truthful mechanism is given for a special case of unit demand auctions (position auctions) in which myopic bidding is not an ex post equilibrium.

Theorem 4.7. *If all players but i are restricted to be consistent then it is a dominant strategy for i to be truthful (even allowing him non consistent strategies).*

Proof. Fix all bidders strategies but i to be consistent, and suppose i is better off using a non consistent strategy in which he gets slot s. Raise both i's value for slot s and his budget to ∞, and assume he follows the consistent strategy with respect to the altered type. Clearly i will still get slot s, and his price at most as he paid by using his non consistent strategy, since the auction outputs a minimal competitive equilibrium. But being truthful is at least as good for i as any other consistent strategy which completes the proof. □

4.3 Independence

So far, the mechanism was stated under the assumption that the valuations and budgets are integers. We begin by relaxing this assumption (assuming that independence still holds). Let

$$\delta = \min_{e_{i,j}, f_i} |\sum e_{i,j} v_i(j) + \sum f_i b_i|$$

where $e_{i,j}, f_i \in \{-1, 0, 1\}$. The value δ is the smallest non negative number that can be reached by substraction and addition of valuations and budgets. As we still assume independence, we have $\delta \neq 0$.

Consider modifying the mechanism by increasing the prices each step by ϵ (for some $\epsilon > 0$). However, even if the players are truthful, different values of ϵ could lead to different allocations or prices. Thus, we define the result of the mechanism as the allocation (and pricing) which is is obtained in the limit $\epsilon \to 0$.

We begin by showing that this is well defined. Fix a sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots \to 0$. As the set of outcomes (including pricing) is compact, there exists a partial limit to the outcomes generated by this sequence. Thus, we only need to show that there is at most one such limit. Indeed, if there are two limits, they differ either in the allocation or in the price of at least one item. Assume first that the outcomes differ in price. Taking ϵ to be this difference multiplied by δ/n, and considering the perturbed envy graph (where i points to j if i envies j's price up to ϵ) shows that the higher price is not obtained. Given the identical pricing, it is easy to show that the same allocation is obtained.

After removing the assumption that the valuations and budgets are integers, one can pick $\epsilon_{i,j}$ at random, such that $\epsilon_{i,j}$ is a discount player i gets if he buys item j. Under these new incentives the valuations and budgets are independent with probability 1.

5 Uniqueness

In this section we prove that any incentive compatible unit demand auction which outputs a competitive equilibrium must output the same outcome as the DGS auction. We will need the following notation. Let M be a truthful auction which for every type $\mathbf{t}$ outputs a competitive equilibrium. We denote by $\mu^M(\mathbf{t})$ the competitive equilibrium which auction M outputs when the profile of types is $\mathbf{t}$. For the DGS auction this graph is denoted as usual by $\mu(\mathbf{t})$. Formally,

Theorem 5.1. *Let M be a truthful unit demand auction with no positive transfers. Then $\mu^M(\mathbf{t}) = \mu(\mathbf{t})$ for every profile of types $\mathbf{t}$.*

Proof. Fix some profile of types $\mathbf{t} = ((v_1, b_1), \ldots, (v_n, b_n))$ and let $\mathbf{q}^M(\mathbf{t})$ and $\mathbf{q}(\mathbf{t})$ be the competitive price vectors in M and DGS respectively when the bidders report $\mathbf{t}$. Similarly let $\mathbf{s}^M(\mathbf{t})$ and $\mathbf{s}(\mathbf{t})$ be the assignments in M and DGS at $\mathbf{t}$. As usual we let $p_i^M(\mathbf{t})$ and $p_i(\mathbf{t})$ i's payment in M and DGS respectively at $\mathbf{t}$. By minimality of the competitive prices in DGS we have $\mathbf{q}^M(\mathbf{t}) \geq \mathbf{q}(\mathbf{t})$.

Claim 5.2. *The set of players that pay zero is identical in DGS and M, i.e. $p_i^M(\mathbf{t}) = 0$ if and only if $p_i(\mathbf{t}) = 0$*

Proof. Assume otherwise and let i be a bidder who paid zero in DGS, has been assigned an item x in M and $q_x^M(\mathbf{t}) > 0$. Since the prices are competitive and $\mathbf{q}^M(\mathbf{t}) \geq \mathbf{q}(\mathbf{t})$ there exists a bidder i' which obtains item x in DGS and (i, i') is an edge in the $\mu(\mathbf{t})$-graph. Again by the competitiveness of the prices it must be that $q_x^M(\mathbf{t}) = q_x(\mathbf{t})$ and (i, i') is red. If i' doesn't receive any item in M, then by the competitiveness of $\mathbf{q}^M(\mathbf{t})$ it must be that $q_x^M(\mathbf{t}) = v_{i'}(x)$ contradicting part 4 of Lemma 3.7. Otherwise, i' received a different item y in M. Following a similar argument we have that $q_y^M(\mathbf{t}) = q_y(\mathbf{t})$ and if bidder i'' obtained y in DGS, then (i', i'') is a red edge in the $\mu^M(\mathbf{t})$-graph. As the number of bidders is finite, some bidder j which received an item in DGS will obtain no item in M - a contradiction. □

Let x be an item which maximizes $q_x^M(\mathbf{t}) - q_x(\mathbf{t})$, and let i be the bidder that is assigned x in DGS. We show:

Lemma 5.3. $\mathbf{s}_i^M(\mathbf{t}) = x$*, i.e. i is assigned x in M.*

Proof. Let $d = q_x^M(\mathbf{t}) - q_x(\mathbf{t})$. Assume that i is not assigned x in M. By Claim 5.2, i gets an item $y \in K$ in M. By the competitiveness of $\mathbf{q}^M(\mathbf{t})$, we have $q_y^M(\mathbf{t}) - q_y(\mathbf{t}) = d$, and there is a red edge (i, i') in the $\mu(\mathbf{t})$-graph where i' is the bidder who obtained y in DGS. The proof continues similarly as the proof of Claim 5.2 while for each item the price difference for each item between the auctions remains d. $\square$

Consider now the following type profile $\tilde{\mathbf{t}}$. For every $i' \neq i$, $\tilde{t}_{i'} = t_{i'}$, and for bidder i let $\tilde{v}_i(x) = \frac{q_x^M(\mathbf{t})+q_x(\mathbf{t})}{2}$, $\tilde{v}_i(y) = 0$ for every $y \neq x$ and $\tilde{b}_i = b_i$.

Claim 5.4. *$\mathbf{s}_i^M(\tilde{\mathbf{t}}) = x$, i.e. i still gets x in M in the type profile $\tilde{\mathbf{t}}$.*

Proof. Since $q(\mathbf{t})$ is also a competitive price vector with respect to $\tilde{t}$, we have $q(\mathbf{t}) \leq q(\tilde{\mathbf{t}})^{DGS}$. Therefore it must be that $s_i(\tilde{\mathbf{t}}) = x$. Since DGS is truthful, $\tilde{q}_x(\tilde{\mathbf{t}}) = q_x(\mathbf{t}) > 0$. Therefore by Claim 5.2 i also gets an item the bidders report $\tilde{\mathbf{t}}$ in M and pays a positive price. Since $\tilde{v}_i(y) > 0$ only for $y = x$ we obtain the result. $\square$

Claim 5.4 leads to a contradiction; If $q_x^M(\mathbf{t}) > q_x^M(\tilde{\mathbf{t}})$ then i is strictly better off reporting $\tilde{t}_i$ when his type is t_i. If $q_x^M(\mathbf{t}) \leq q_x^M(\tilde{\mathbf{t}})$ then since $\tilde{v}_i(x) = \frac{q_x^M(\mathbf{t})+q_x(\mathbf{t})}{2} < q_x^M(\mathbf{t})$, i pays more then he values x in $\tilde{t}_i$.

$\square$

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