Supplementary Material for Monotonicity and Implementability

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1 Domains with Convex Closure

Saks and Yu (2005) proved that if D is convex then every monotone deterministic allocation rule is implementable. We prove in this appendix the following generalization of their result:

Theorem 1 *Every domain with a convex closure is a proper monotonicity domain.*

1.1 Preparations

First we recall the definitions of monotonicity and cyclic monotonicity. An allocation rule f is called *monotone* if

$$\langle f(v) - f(w), v - w \rangle \geq 0 \quad \text{for every } v, w \in D, \tag{1}$$

and f is called *cyclically monotone* if for every $k \geq 2$, for every k vectors in D (not necessarily distinct), $v_1, v_2, \ldots, v_k$ the following holds:

$$\sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i) \rangle \geq 0, \tag{2}$$

where v_{k+1} is defined to be v_1. By taking $k = 2$ in (2) it can be seen that every cyclically monotone allocation rule is monotone.

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Let $f : D \to \bar{Z}(A)$ be monotone and finite-valued, where D is an arbitrary set. Let $y^1, \ldots, y^m \in R^A$ be the distinct values of f. That is, for every $v \in D$ there exists $1 \leq j \leq m$ such that $f(v) = y^j$, and every y^j is attained at some valuation. If $m > 1$, for $j \neq k$ define:

$$\delta(j,k) = \delta_{D,f}(j,k) = \inf_{v \in D, f(v)=y^j} \langle v, y^j - y^k \rangle. \tag{3}$$

If $w \in D$ satisfies $f(w) = y^k$ then by monotonicity, $\langle v, y^j - y^k \rangle \geq \langle w, y^j - y^k \rangle$. Therefore $\delta(j,k) > -\infty$. Furthermore:

$$\delta(j,k) \geq \sup_{v \in D, f(v)=y^k} \langle v, y^j - y^k \rangle = -\delta(k,j).$$

Hence,

$$\delta(j,k) + \delta(k,j) \geq 0, \quad \forall j \neq k. \tag{4}$$

As (2) can be written as

$$\sum_{i=1}^{k} \langle v_i, f(v_i) - f(v_{i-1}) \rangle \geq 0, \tag{5}$$

where v_0 is defined to be v_k, the following useful lemma has been noted by many authors (see e.g. Heydenreich et al. (2007); Saks and Yu (2005)):

Lemma 2 *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone.*

a. f is cyclically monotone if and only if for every sequence $j_1, j_2, \ldots, j_k$, $k \geq 2$, such that $j_s \neq j_{s+1}$ for $1 \leq s < k$ the following holds:

$$\sum_{i=1}^{k} \delta(j_i, j_{i+1}) \geq 0, \tag{6}$$

where j_{k+1} is defined to be j_1.

b. If in addition to the monotonicity $\delta(j,k) + \delta(k,j) = 0$ for every $j \neq k$, then f is cyclically monotone if and only if the inequalities (6) are satisfied as equalities.

For every j let:

$$D_j = \{v \in D | \ \langle v, y^j - y^k \rangle \geq \delta(j,k) \quad \forall k, \ k \neq j\}.$$

Obviously, $f(v) = y^j$ implies $v \in D_j$. Hence, $D = \cup_{j=1}^m D_j$.

The following sufficient condition will be useful:

Lemma 3 *Let $f : D \to \bar{Z}(A)$ be finite-valued and monotone. If $\cap_{j=1}^m D_j \neq \emptyset$ then f is cyclically monotone.*

Proof: Let $v \in D$ be in the intersection. Hence $\langle v, y^j - y^k \rangle \geq \delta(j,k)$ for all $j \neq k$. We claim that

$$\langle v, y^j - y^k \rangle = \delta(j,k) \quad \text{for all } j \neq k. \tag{7}$$

Indeed, $v \in D_j$ implies $\langle v, y^j - y^k \rangle \geq \delta(j,k)$, and $v \in D_k$ implies $\langle v, y^k - y^j \rangle \geq \delta(k,j)$. Therefore, from (4) we obtain (7) . By plugging (7) in (6) it follows that (6) is satisfied with equality for every sequence of indices, and hence f is cyclically monotone.■

We next show that in order to prove that a set is a proper monotonicity domain it suffices to prove that its closure is a proper monotonicity domain. For a domain D we denote its closure by $cl(D)$.

Lemma 4 *If $cl(D)$ is a proper monotonicity domain so is D.*

Proof: Suppose $cl(D)$ is a proper monotonicity domain, and let $f : D \to \bar{Z}(A)$ be a finite-valued monotone function on D. Extend f to $cl(D)$ as follows: For every $v \in cl(D) \setminus D$ there exists a sequence v_n, $n \geq 1$ in D such that $v_n \to v$. For some j there exists an infinite numbers of indices n such that $f(v_n) = y^j$. Hence for every $v \in cl(D) \setminus D$ there exists j and a sequence $v_n \in D$ such that $v_n \to v$ and $f(v_n) = y^j$ for every $n \geq 1$. Let $f(v) = y^j$ for such arbitrary j. It is easily verified that the extension of f is monotone on $cl(D)$. Therefore it is cyclically monotone on $cl(D)$, and therefore f is cyclically monotone on D.■

We will use a characterization of cyclically monotone functions that can easily be derived from Section 24 in Rockafellar (1970).

Theorem 5 (Rockafellar) *Let $D \subseteq R^A$ be a convex and non-empty subset of valuations, and let $f : D \to \bar{Z}(A)$.*

a. f is cyclically monotone on D if and only if there exists a real-valued function U on D such that[1]

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle, \quad \forall v_1, v_2 \in D. \tag{8}$$

b. If each of the functions $U_1, U_2 : D \to R$ satisfies (8), then the functions differ by a constant. That is, there exists a real number α such that

$$U_1(v) = U_2(v) + \alpha \quad \forall v \in D. \tag{9}$$

c. Suppose that $U : D \to R$ satisfies (8), and let $v_1 \neq v_2 \in D$. Then, the real-valued function

$$\phi(t) = \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle \tag{10}$$

[1] $U(v)$ can be interpreted as the utility function of the agent when her valuation is v.

defined for every $t \in [0,1]$ is non-decreasing, and:

$$U(v_2) - U(v_1) = \int_0^1 \phi(t)dt, \tag{11}$$

where the integral is computed in the sense of Riemann.[2]

The main tool in proving Theorem 1 is the following:

Theorem 6 *Let $D = H_1 \cup H_2$ be a closed convex set, where each H_i is closed convex and non-empty. Let $f : D \to \bar{Z}(A)$ be monotone (not necessary finite-valued). If f is cyclically monotone on every H_i then f is cyclically monotone on D.*

Proof: Because D and the sets H_i are closed, $H_1 \cap H_2 \neq \emptyset$. Let v^* be a fixed valuation in $H_1 \cap H_2$. By Part a of Theorem 5, there exists U_1 on H_1 that satisfies (8) on H_1. By adding a constant, we can choose U_1 such that $U_1(v^*) = 0$. Similarly there exists $U_2 : H_2 \to R$ that satisfies (8) on H_2 and $U_2(v^*) = 0$. By Part b of Theorem 5, $U_1 = U_2$ on $H_1 \cap H_2$. Hence we can define a function U on D by $U(v) = U_i(v)$ for $v \in H_i$. In order to show that f is cyclically monotone on D, it suffices by Part a to show that (8) is satisfied by U on D. Let then $v_1 \neq v_2$ in D. Obviously we can consider only the case $v_1 \in H_1 \setminus H_2$, $v_2 \in H_2 \setminus H_1$. Because H_1, H_2 and D are closed and $v_1 \in H_1 \setminus H_2$ and $v_2 \in H_2 \setminus H_1$, the interval (v_1, v_2) intersects $H_1 \cap H_2$, say $w = v_1 + s(v_2 - v_1)$, $0 < s < 1$ is a valuation at the intersection. By applying Part c of Theorem 5 to v_1, w in H_1, and by a simple change of variables we get:

$$U(w) - U(v_1) = \int_0^s \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt,$$

and similarly

$$U(v_2) - U(w) = \int_s^1 \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt.$$

Therefore:

$$U(v_2) - U(v_1) = \int_0^1 \langle f(v_1 + t(v_2 - v_1)), v_2 - v_1 \rangle dt.$$

By the monotonicity of f, the integrand is non-decreasing in t, and therefore the integral is greater or equals the value of the integrand at $t = 0$. Hence,

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle . \blacksquare$$

[2] A non decreasing function is Riemann integrable. It is also Borel measurable and therefore its Riemann integral equals its Lebesgue integral.

1.2 Proof of Theorem 1:

We first show that it suffices to prove that every compact convex set is a proper monotonicity domain. Let D be a set such that $cl(D)$ is convex. By Lemma 4 it suffices to prove that $cl(D)$ is a proper monotonicity domain.

Assume the result holds for every compact convex set, and assume in negation that $f : cl(D) \to \bar{Z}(A)$ is a finite-valued monotone randomized allocation rule, which is not cyclically monotone. Therefore there exist $v_1, v_2, \cdots, v_k$ in $cl(D)$ that contradict (2). Let K be the convex hull of these valuations, then f is finite-valued and monotone on K and it is not cyclically monotone, contradicting our assumption that the assertion holds for compact convex sets.

We prove the theorem for compact convex sets by a double induction process. The first induction is on the number of distinct values, $m(D, f)$ of f on D. If $m(D, f) = 1$ then obviously f is cyclically monotone. Let $m > 1$, and assume we have already proven that for every compact convex D and for every monotone randomized allocation rule $f : D \to \bar{Z}(A)$ with $m(f, D) < m$, f is cyclically monotone on D. We proceed to prove it for every $m(D, f) = m$.

For every (D, f) with $f(D) = \{y^1, \cdots y^m\}$ let $r(D, f)$ be the maximal number r, $1 \leq r \leq m$ for which for every set F of r distinct values in $\{1, \ldots, m\}$, the intersection $\cap_{j \in F} D_j \neq \emptyset$. We prove our result by induction on $r(D, f)$. Let then $r(D, f) = 1$. Since $m > 1$ there exists $j \neq k$ such that $D_j \cap D_k = \emptyset$. Since D_j and D_k are compact and convex we can strongly separate them. That is, there exists $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in D_j, \forall w \in D_k.$$

Denote $H_1 = \{v \in D | \langle v, y \rangle \leq \alpha\}$, $H_2 = \{v \in D | \langle v, y \rangle \geq \alpha\}$. On each H_i the function f takes at most $m - 1$ values, and therefore by the first induction hypothesis f is cyclically monotone on each H_i. By Theorem 6 f is cyclically monotone on D. Suppose the theorem is proved for $1, \ldots, r-1$, $2 \leq r \leq m$. We now prove it for $r(D, f) = r$. If $r = m$ the result follows from Lemma 3. If $r < m$ there exists a set of indices of cardinality $r + 1$, which w.l.o.g. we take to be $\{1, \ldots, r+1\}$, such that $\cap_{j=1}^{r} D_j \neq \emptyset$, and $\cap_{j=1}^{r+1} D_j = \emptyset$. The convex compact sets $\cap_{j=1}^{r} D_j$ and D_{r+1} must be strongly separated. That is, there exists $0 \neq y \in R^A$ and $\alpha \in R$ such that

$$\langle v, y \rangle < \alpha < \langle w, y \rangle \quad \forall v \in \cap_{j=1}^{r} D_j, \forall w \in D_{r+1}.$$

Let $H_1 = \{v \in D | \langle v, y \rangle \leq \alpha\}$, $H_2 = \{v \in D | \langle v, y \rangle \geq \alpha\}$. On H_1 the function f does not take the value y^{r+1} and therefore by our first induction hypothesis f is cyclically monotone. On H_2, if $m(H_2, f) < m$ then f is implementable on H_2 by the first induction hypothesis.

Suppose $m(H_2, f) = m$. Since $H_2 \subseteq D$, $\delta_{H_2,f}(j,k) \geq \delta_{D,f}(j,k)$ for every $j \neq k$. Therefore, for every j, $H_{2_j} \subseteq D_j$, where $H_{2_j} = \{v \in H_2 | \ \langle v, y^j - y^k \rangle \geq \delta_{H_2,f}(j,k)\}$. Hence, $\cap_{j=1}^r H_{2_j} \subseteq H_2 \cap (\cap_{j=1}^r D_j) = \emptyset$ implying $r(H_2, f) < r$. Therefore by our second induction hypothesis f is cyclically monotone on H_2. Hence f is cyclically monotone on D by Theorem 6.■

1.3 A Note on General Monotone Allocation Rules

The definitions of monotonicity and cyclic monotonicity are not restricted to functions that take only sub-probability values. Hence, every function, $f : D \to R^A$, that satisfies (1) ((2)) is called *monotone* (*cyclically monotone*). Such general functions can be used, e.g., in models with divisible goods. It is therefore interesting to note that without any change in the proofs Theorem 1 holds for such functions. Therefore the following result holds:

Theorem 7 *Let $D \subseteq R^A$ be a domain with a convex closure. Every finite-valued monotone function $f : D \to R^A$ is cyclically monotone.*

References

B. Heydenreich, R. Müller, M. Uetz, and R. Vohra. On revenue equivalence in truthful mechanisms. In *Computational Social Systems and the Internet*, Dagstuhl Seminar Proceedings, 2007.

R.T. Rockafellar. *Convex analysis*. Princeton Univ. Press, 1970.

M. E. Saks and L. Yu. Weak monotonicity suffices for truthfulness on convex domains. In *Proceedings of the 6th ACM conference on Electronic commerce*, pages 286–293, 2005.